

# Choosability with separation of complete graphs and minimal abundant packings

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## Abstract

For a graph  $G$  and a positive integer  $c$ , let  $\chi_l(G, c)$  be the minimum value of  $k$  such that one can properly color the vertices of  $G$  from any lists  $L(v)$  such that  $|L(v)| = k$  for all  $v \in V(G)$  and  $|L(u) \cap L(v)| \leq c$  for all  $uv \in E(G)$ . Kratochvíl et al. asked to determine  $\lim_{n \rightarrow \infty} \chi_l(K_n, c)/\sqrt{cn}$ , if exists. We prove that the limit exists and equals 1. We also find the exact value of  $\chi_l(K_n, c)$  for infinitely many values of  $n$ .

Given a graph  $G(V, E)$ , a *list*  $L$  for  $G$  is an assignment to every  $v \in V(G)$ , a set  $L(v)$  of colors that may be used for the coloring of  $v$ . We say that  $G$  is  $L$ -colorable, if there exists a proper coloring  $f$  of the vertices of  $G$  from  $L$ , i.e. if  $f(v) \in L(v)$  for all  $v \in V(G)$  and  $f(u) \neq f(v)$  for all  $uv \in E$ . An extensively studied parameter is the *list chromatic number* of  $G$ ,  $\chi_l(G)$ , which is the least  $k$  such that  $G$  is  $L$ -colorable, whenever  $|L(v)| = k$  for all  $v \in V(G)$ .

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We say that a list  $L$  for a graph  $G$  is a  $(k, c)$ -list if  $|L(v)| = k$  for all  $v \in V(G)$  and  $|L(u) \cap L(v)| \leq c$  for all  $uv \in E(G)$ . Kratochvíl et al. [5] introduced  $\chi_l(G, c)$ , the least  $k$  such that  $G$  is  $L$ -colorable from each  $(k, c)$ -list  $L$ . Among other results, they showed that  $\sqrt{\frac{cn}{2}} \leq \chi_l(K_n, c) \leq \sqrt{2ecn}$ , where  $K_n$  is the complete graph on  $n$  vertices. They also asked whether the limit  $\lim_{n \rightarrow \infty} \chi_l(K_n, c)/\sqrt{cn}$  exists. We prove that the limit is 1. We also find the exact value of  $\chi_l(K_n, c)$  for infinitely many values of  $n$ .

## 1 Upper Bound

We start by citing two known facts.

**Lemma 1** (Johnson's bound). [4] *Let  $E_1, \dots, E_m$  be sets such that  $|E_i| \geq k$  and  $|E_i \cap E_j| \leq c$ . Then  $|\bigcup_{i=1}^m E_i| \geq \frac{mk^2}{mc+k-c}$ .*

For the complete graph  $K_n$  and a list  $L$ , the *vertex-color adjacency graph*,  $F = F(V(K_n), U_L)$ , is the bipartite graph whose partite sets are  $V(K_n)$  and  $U_L = \bigcup_{v \in V(K_n)} L(v)$  with  $u \in V(K_n)$  being adjacent to  $\alpha \in U_L$  if and only if  $\alpha \in L(u)$ .

**Observation 2** (Vizing). [6] *For every list assignment  $L$  for  $K_n$ ,  $K_n$  has an  $L$ -coloring if and only if the vertex-color adjacency graph  $F(V(K_n), U_L)$  has a matching saturating  $V(K_n)$ .*

**Lemma 3.** *Let  $c \geq 1$ . If  $\mathcal{L}(V, E)$  is a  $q$ -uniform hypergraph such that  $|E| = q + 2$  and  $|e \cap e'| \leq c - 1$ , then  $|V| \geq \frac{q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1}}{c}$ .*

*Proof.* Let  $d_v$  be the degree of vertex  $v$ , then we have

$$\sum_v \binom{d_v}{2} = \sum_{e, e' \in E} |e \cap e'| \leq (c-1) \binom{q+2}{2} \quad (1)$$

$$\sum_v d_v = \sum_e |e| = (q+2)(q) \quad (2)$$

□

Let  $t \geq 1$  be an integer. We multiply (1) by  $\frac{-1}{\binom{t}{2}}$ , (2) by  $\frac{2}{t}$  and add them up. We also note the fact that  $\frac{2}{t}d - \frac{1}{\binom{t}{2}}\binom{d}{2} \leq 1$  when  $t \in \{2, 3, \dots\}$  and  $d \in \{1, 2, \dots\}$ . We now choose  $t = c + 1$  and we have

$$\sum_v 1 \geq \sum_v \frac{2}{c+1}d_v - \frac{1}{\binom{c+1}{2}}\binom{d_v}{2} \geq \frac{2}{c+1}q(q+2) - \frac{1}{\binom{c+1}{2}}(c-1)\binom{q+2}{2} = \frac{q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1}}{c}$$

$$\text{Hence } |V| \geq \frac{q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1}}{c}$$

**Lemma 4.** *Let  $L$  be a list assignment such that  $|L(v)| \geq q + 1$ . Then  $F(V(K_n), U_L)$  has a matching saturating  $V(K_n)$  if  $n \leq \frac{q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1}}{c}$ .*

*Proof.* We need to show that Hall's condition holds in  $F$ , that is,  $|S| \leq |N(S)|$ , for all  $S \subseteq V$ . For this we consider the subgraph  $F_S$  induced by vertices of  $S$  and  $N(S)$

*Case 1 :*  $\deg_{F_S}(\alpha) \leq q + 1$ , for all  $\alpha \in N(S)$ .

Counting edges in  $F_S$  we have

$$|S|(q + 1) = \sum_{v \in S} |L(v)| = \sum_{\alpha \in N(S)} \deg_{F_S}(\alpha) \leq |N(S)|(q + 1),$$

which implies  $|S| \leq |N(S)|$ .

*Case 2:*  $\deg_{F_S}(\alpha) \geq q + 2$ , for some  $\alpha \in N(S)$ .

Suppose  $\alpha \in L(v_1) \cup L(v_2) \cup \dots \cup L(v_{q+2})$ , where  $v_1, \dots, v_{q+2} \in S$ . Consider the sets  $L'_i = L(v_i) \setminus \{\alpha\}$ . Then  $|L'_i| \geq q$  and  $|L'_i \cap L'_j| \leq c - 1$ , for all  $1 \leq i, j \leq q + 2, i \neq j$ . We now consider a hypergraph  $\mathcal{L}$  with  $L'_i$  as its edges. By Lemma 3,

$$|\cup_i L(v_i)| \geq 1 + \frac{q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1}}{c}$$

Now if Hall's condition fails to hold, then  $|S| > |N(S)|$ , and thus  $n \geq |S| > |N(S)| \geq |\cup_i L(v_i)| \geq 1 + \frac{q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1}}{c}$ , which is a contradiction.  $\square$

**Remark:** One could also use Lemma 1 in Case 2 of the above Lemma to obtain a slightly weaker upper bound of  $\frac{q^2(q+2)}{c(q+1)-1}$ .

Observation 2 and Lemma 4 now yield the following Proposition.

**Proposition 5.**  $\chi_l(K_n, c) \leq q + 1$  for  $n \leq \frac{q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1}}{c}$ .

## 2 Lower Bound

In this section we will obtain a lower bound on  $\chi_l(K_n, c)$  and then use it to yield the main theorem stated below

**Theorem 6.** *Let  $c \geq 1$ , then*

(i)  $\lim_{n \rightarrow \infty} \chi_l(K_n, c) / \sqrt{cn} = 1$

(ii) *If  $q$  is a prime power,  $c < q - 1$  and  $c$  divides  $q - 1$ , then  $\chi_l(K_n, c) = q + 1$ , for all  $n \in [\frac{q^2-1}{c} + 2, \frac{q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1}}{c}]$ .*

For this we first show the construction based on [1] and then use it to get a lower bound on  $\chi_l(K_n, c)$ .

Let  $q$  be a prime power and  $c$  an integer such that  $c < q - 1$  and  $c$  divides  $q - 1$ . Let  $\mathbf{F}$  be the  $q$ -element finite field  $\mathbf{GF}(q)$  and let  $h$  be an element of order  $c$  in the multiplicative group  $\mathbf{F} \setminus \{0\}$ . Set  $H = \{1, h, h^2, \dots, h^{c-1}\}$ . Then  $H$  is a  $c$ -element subgroup of  $\mathbf{F} \setminus \{0\}$ . Let  $(a, b), (a', b') \in \mathbf{F} \times \mathbf{F} \setminus \{(0, 0)\}$ . We say that  $(a, b) \sim (a', b')$ , if there exists  $h^\alpha \in H$  such that  $a' = h^\alpha a$  and  $b' = h^\alpha b$ . Note that  $\sim$  is an equivalence relation and each equivalence class is

a collection of  $c$  elements in  $\mathbf{F} \times \mathbf{F} \setminus \{(0, 0)\}$ . Hence there are  $\frac{q^2-1}{c}$  equivalence classes. The equivalence class containing  $(a, b)$  will be denoted by  $\langle a, b \rangle$ .

Consider the set  $L\langle a, b \rangle = \{\langle x, y \rangle : ax + by \in H\}$ . Since  $H$  is a group,  $ax + by \in H$  implies  $(h'a)x + (h'b)y \in H$ , for all  $h' \in H$ . Hence  $L\langle a, b \rangle$  is well-defined.

**Claim 1:** Let  $(a, b), (a', b') \in \mathbf{F} \times \mathbf{F} \setminus \{(0, 0)\}$ . Then  $|L\langle a, b \rangle| = q$ . Moreover, if  $(a, b) \approx (a', b')$ , then either  $|L\langle a, b \rangle \cap L\langle a', b' \rangle| = c$  or  $|L\langle a, b \rangle \cap L\langle a', b' \rangle| = 0$ .

Proof: Let  $(a, b) \in \mathbf{F} \times \mathbf{F} \setminus \{(0, 0)\}$ . By symmetry we assume  $b \neq 0$ . Then for any given  $x$  and  $h^\alpha$ , there is a unique solution of  $ax + by = h^\alpha$ . Hence there are exactly  $qc$  solutions. These solutions come in equivalence classes and hence  $|L\langle a, b \rangle| = q$ .

Now consider  $(a, b) \approx (a', b')$ . Then for given  $\alpha$  and  $\beta$ , if the system of equations

$$\begin{aligned} ax + by &= h^\alpha \\ a'x + b'y &= h^\beta \end{aligned}$$

has a solution, then it has a unique solution, since  $\det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \neq 0$ . Hence for  $c^2$  values of  $\alpha$  and  $\beta$  there are  $c^2$  possible solutions. Since the solutions come in equivalence classes, either  $|L\langle a, b \rangle \cap L\langle a', b' \rangle| = c$  or  $|L\langle a, b \rangle \cap L\langle a', b' \rangle| = 0$ .

Applying Claim 1 we obtain the following theorem.

**Theorem 7** (Furedi). [1] *Let  $\mathcal{L}(V, E)$  be a hypergraph with the vertex set  $V = \{\langle a, b \rangle : a, b \in \mathbf{F}, (a, b) \neq (0, 0)\}$  and the edge set  $E = \{L\langle a, b \rangle : a, b \in \mathbf{F}, (a, b) \neq (0, 0)\}$ . Then  $\mathcal{L}$*

(i) *is a  $q$ -uniform hypergraph*

(ii) *has  $\frac{q^2-1}{c}$  vertices*

(iii) *has  $\frac{q^2-1}{c}$  edges such that every two edges intersect in at most  $c$  vertices.*

**Claim 2:** If  $V_m = \{\langle x, y \rangle : y = mx\}$ , then  $|V_m| = \frac{q-1}{c}$  and  $|V_m \cap L\langle a, b \rangle| \leq 1$ , for each  $\langle a, b \rangle \in V$ .

Proof: Given  $m$ ,  $y = mx$  has  $q$  solutions in  $\mathbf{F} \times \mathbf{F}$ . Since the solutions come in equivalence classes  $y = mx$  has  $\frac{q-1}{c}$  solutions in  $V$ , where  $V$  is the vertex set of the hypergraph  $\mathcal{L}$  from Theorem 7.

To prove the next claim, we might see the hypergraph  $\mathcal{L}$  arising out of the affine plane geometry  $AG(2, q)$  by combining  $c$  parallel lines of the form  $ax + by = h^\alpha$  for  $c$  values of  $\alpha$  to get the set  $\langle a, b \rangle$ . Now in  $AG(2, q)$  every two lines intersect in at most 1 point in  $\mathbf{F} \times \mathbf{F}$ . Hence for each  $\alpha$  the line  $ax + by = h^\alpha$  meets the line  $y = mx$  in at most 1 point in  $V(\mathcal{L})$  and thus  $|V_m \cap L\langle a, b \rangle| \leq 1$ .  $\square$

**Proposition 8.** *Let  $q$  be a prime power. If  $c < q-1$  and  $c$  divides  $q-1$ , then  $\chi_l(K_n, c) \geq q+1$  for  $n \geq \frac{q^2-1}{c} + 2$ , that is  $\chi_l(K_n, c) \geq \sqrt{c(n-2)+1} + 1$ .*

*Proof.* Consider the hypergraph  $\mathcal{H}(V', E')$  with vertex set  $V' = V \cup \{x\}$ , where  $x \notin V$  and the edge set  $E' = E \cup \{\{x\} \cup V_1 \cup V_2 \cup \dots \cup V_c, \{x\} \cup V_{c+1} \cup V_{c+2} \cup \dots \cup V_{2c}, \dots\}$ , where  $V, E, V_i$  sets considered in Claim 2. By Claim 2 and Theorem 7,  $|\{x\} \cup V_1 \cup V_2 \cup \dots \cup V_c| = 1 + c \frac{q-1}{c} = q$  and every two edges intersect in at most  $c$  vertices. Thus  $\mathcal{H}(V', E')$  is a  $q$ -uniform hypergraph such that  $|V'| = \frac{q^2-1}{c} + 1$  and  $|E'| > |V'|$  such that  $|e \cap e'| \leq c$  for every  $e, e' \in E'$ .

Let  $n = \frac{q^2-1}{c} + 2$ . Consider the hypergraph  $\mathcal{H}$  constructed above. Let  $f : V(K_n) \rightarrow E'$  be a bijective mapping. For every  $v \in V(K_n)$  we let its list be  $L(v) = f(v)$ . Then  $L$  is a  $(q, c)$ -list assignment in which the total number of colors  $|V'| < n$ . Hence there is no proper coloring of  $K_n$  with this list assignment. Hence  $\chi_l(K_n, c) \geq q + 1$  for  $n = \frac{q^2-1}{c} + 2$  and thus  $\chi_l(K_n, c) \geq q + 1$  for  $n \geq \frac{q^2-1}{c} + 2$ .  $\square$

We shall use the following lemma to give a general lower bound for any  $n$ .

**Lemma 9.** [3] *Let  $c \geq 1$ ,  $n$  be sufficiently large. Then the interval  $[\sqrt{c(n-2)+1} + 1 - n^{1/3}, \sqrt{c(n-2)+1} + 1]$  contains a prime  $q$  such that  $c$  divides  $q - 1$ .*

**Proposition 10.** *Let  $c \geq 1$ . Then for every sufficiently large positive integer  $n$ ,  $\chi_l(K_n, c) \geq \lfloor \sqrt{c(n-2)+1} + 1 \rfloor - n^{1/3}$ .*

*Proof.* Given a sufficiently large  $n$ , consider the interval  $[\sqrt{c(n-2)+1} + 1 - n^{1/3}, \sqrt{c(n-2)+1} + 1]$ . By Lemma 9, this interval contains a prime  $q$  such that  $c$  divides  $q - 1$ . Let  $n' = \frac{(q^2-1)}{c} + 2$ . By Proposition 8,  $\chi_l(K_{n'}, c) \geq \sqrt{c(n-2)+1} + 1 = q + 1$ . Hence  $\chi_l(K_n, c) \geq \chi_l(K_{n'}, c) \geq q + 1 \geq \lfloor \sqrt{c(n-2)+1} + 1 \rfloor - n^{1/3}$ .  $\square$

Propositions 5, 10 and 8 now imply Theorem 6.

For a fixed  $c \geq 1$ , one might be interested in knowing what is the maximum value of  $\chi_l(G, c)$  over all  $n$ -vertex graphs  $G$ . Note that if  $H$  is an induced subgraph of  $G$ , then  $\chi_l(H, c) \leq \chi_l(G, c)$ , but this may not hold true for non-induced subgraphs. We have the following conjecture.

**Conjecture 11.** *If  $c, n \geq 1$  and  $G$  is an  $n$ -vertex graph, then  $\chi_l(G, c) \leq \chi_l(K_n, c)$ .*

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